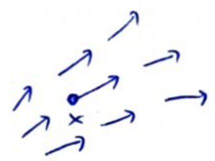
Math 132: Differential Topology

§ Vector fields and the Poincaré-Hopf theorem

A vector field on a manifold M is a smooth section $v : M \to TM$,
i.e. a smooth assignment of a tangent vector $v(x) \in T_xM$ at each point $x \in M$.

If $v(x) \neq 0$ at some point $x \in M$, then v is nearly constant in magnitude and direction near x:



However, if $v(x) = 0$, v can behave interestingly around such zeros.

Ex Let's say M is a surface.
Here are some possible behaviors of v around its zeros:

sink source circulation spiral saddle

We'll see that the zeros of v encode some information on the topology of the manifold M!

Suppose v is a vector field on $\mathbb{R}^m$ that has an isolated zero at the origin $0 \in \mathbb{R}^m$.

Then, for any small enough radius $\varepsilon > 0$, v has no zeros except at the origin inside the sphere S^{m-1}_ε of radius ε around the origin.

Define the index of v at 0, $\mathrm{ind}_0(v)$, to be the degree of the directional map

$$S^{m-1}_\varepsilon \longrightarrow S^{m-1}$$
$$x \longmapsto \frac{v(x)}{|v(x)|}$$

Ex In the two-dimensional case, $\mathrm{ind}_x(v)$ simply counts the number of times v rotates (in the counterclockwise direction) as we walk counterclockwise around the circle.

e.g.

index: $+1$ $\quad +1 \quad +1 \quad +1 \quad -1 \quad +2$

More generally, if x is an isolated zero of a vector field v on M, then we can define $\mathrm{ind}_x(v)$ to be the index of the pushforward vector field $\varphi^{-1}_*(v)$ at 0, where $\varphi : U \to M$, $0 \mapsto x$ is a local parametrization.

Here's the main theorem of this lecture:

Thm (Poincaré–Hopf index theorem)

If v is a smooth vector field on a compact, oriented manifold M with only finitely many zeros, then $\chi(M) = \sum_{v(x)=0} \mathrm{ind}_x(v)$.

sum over zeros

Cor (Hairy ball theorem)

There's no nowhere vanishing vector field on S^2, since $\chi(S^2) = 2 \neq 0$.

"You can't comb a hairy ball without leaving a bald spot."

proof of Poincaré–Hopf)

The idea is that a vector field determines a flow

$f_t : M \to M$ that is homotopic to $f_0 =$ identity,

and tangent to v at time zero.

(Such family of maps can be obtained by solving the ODE given by the vector field. Another way is to simply take $f_t(x) = \pi(x + t\,v(x))$, where π is the projection map from the ε-neighborhood theorem.)

For small $t \neq 0$, fixed points of f_t correspond exactly to zeros of v,

so it suffices to prove that $\operatorname{ind}_x(v) = L_x(f_t)$.

local Lefschetz number

This is a local problem, so we may work in $\mathbb{R}^m$.

From basic calculus, there is a smooth vector-valued function $r(t,x)$ such that

$$\begin{aligned} f_t(x) &= f_0(x) + t\, f_0'(x) + t^2 r(t,x) \\ &= f_0(x) + t\, v(x) + t^2 r(t,x), \end{aligned}$$

so $$\frac{f_t(x) - x}{|f_t(x) - x|} = \frac{v(x) + t\, r(t,x)}{|v(x) + t\, r(t,x)|} : S_\varepsilon^{m-1} \to S^{m-1}$$

is homotopic to $\frac{v(x)}{|v(x)|}$, whose degree is $\operatorname{ind}(v)$.

On the other hand, the local Lefschetz number $L_x(f_t)$ is the degree of the map $\partial B_\varepsilon^m \to S^{m-1}$, $z \mapsto \frac{f_t(z) - z}{|f_t(z) - z|}$. (For Lefschetz fixed points, simple linear algebra shows that this agrees with the previous definition, since $\deg\left(z \mapsto \frac{(A-I)z}{|(A-I)z|}\right) = \operatorname{sign} \det(A-I)$.

Homotopy invariance follows from boundary theorem:

degree on outer ∂ = Sum of degrees on the inner ∂)

<u>Cor</u> $\chi(M) = I(M, M)$, where M is considered as the 0-section $M \subset TM$. ∎